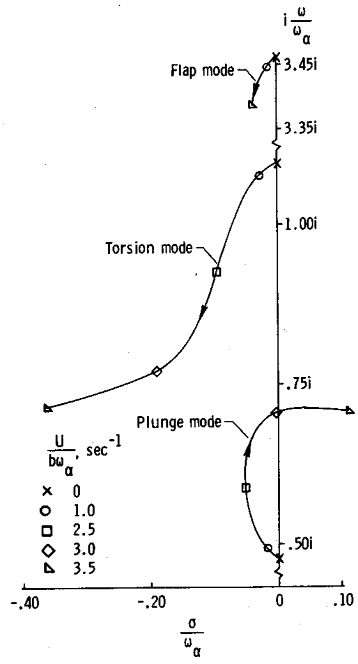

Fig. 2 Locus of roots of a three-degree-of-freedom section vs $U/b\omega_\alpha$ in incompressible flow. $\mu=40$, $a=-0.4$, $c=0.6$, $x_\alpha=0.2$, $r_\alpha{}^2=0.25$, $x_\beta=0.0125$ $r_\beta{}^2=0.00625$, $\omega_\alpha=100$ rad/sec, $\omega_h=50$ rad/sec, $\omega_\beta=300$ rad/sec.

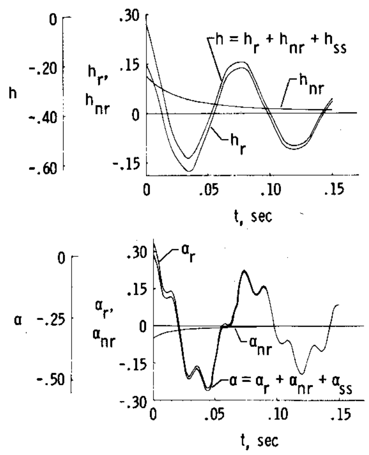

Fig. 3 Rational, nonrational, and total plunge and pitch due to a step command to the flap. $M=0$; $h_{ss}=-0.39$; $\alpha_{ss}=-0.28$.

Transient airfoil motions may then be calculated by evaluating the inversion integral along the deformed contour in Fig. 1. Figure 3 shows the calculated plunge and rotation response of the section of Fig. 2 due to a step input command to the flap. The responses are composed of rational portions (due to the poles contained within the contour) and nonrational portions (due to the integral along the branch cut). The nonrational portions do not participate in the oscillatory response characteristic of fluttering airfoils.

Generalized Compressible Unsteady Aerodynamics

It is natural to extend the technique of Laplace transformation of the study of loads due to arbitrary airfoil motions in compressible flow. The transformed three-dimensional linearized partial differential equation of unsteady aerodynamics is

$$\Phi_{xx}+\Phi_{yy}+\Phi_{zz}-(s^2/a^2)\Phi-(2Ms/a)\Phi_x-M^2\Phi_{xx}=f(x,y,z,0) \tag{8}$$

where $\Phi(x,y,z,s)=\ell[\varphi(x,y,z,t)$ and

$$f(x,y,z,0)=-(s/a^2)\varphi(x,y,z,0)-(1/a^2)\varphi_t(x,y,z,0)-(2M/a)\varphi_x(x,y,z,0) \tag{9}$$

The transformed boundary condition is

$$\ell[w_a(x,y,t)]=\Phi_z|_{z=0}=\left(s+U\frac{\partial}{\partial x}\right)\ell[z_a(x,y,t)]-z_a(x,y,0) \tag{10}$$

The terms $f(x,y,z,0)$ in Eq. (8) and $z_a(x,y,0)$ in Eq. (10) are initial conditions resulting from the Laplace integral transform. Since Eq. (8) is linear, the solution can be obtained as a superposition of solutions

$$\Phi(x,y,z,s)=\Phi_1(x,y,z,s)+\Phi_2(x,y,z,s) \tag{11}$$

where Φ_2 is regarded as a known function chosen to satisfy Eq. (8) subject to the boundary condition

$$\left[\frac{\partial}{\partial z}\Phi_2\right]_{z=0}=-z_a(x,y,0) \tag{12}$$

Then the solution Φ_1 satisfies a boundary-value problem which is formally identical to that resulting from the assumption of simple harmonic motion with the replacement of $i\omega$ by $s=\sigma+i\omega$. Thus, digital programs that calculate airloads due to simple harmonic motions may be modified in a fairly straightforward manner to calculate airloads due to arbitrary motions corresponding to the solution Φ_1.

Edwards[1] applies this technique to Garrick and Rubinow's[6] solution for two-dimensional supersonic flow. Airloads for complex values of $s=\sigma+i\omega$ and root loci of aeroelastic modes as a function of Mach number are presented in Ref. 1.

The loads resulting from the Φ_1 solution are linear with respect to the transformed airfoil motions, while the loads resulting from the Φ_2 solution are linear with respect to the initial conditions of the motion. Thus, the resulting airloads will take the form $A_1X(s)+A_2x(0)$ where $X(s)$ and $x(0)$ are $n\times 1$ vectors of normal coordinates. As a consequence, only the airloads corresponding to the Φ_1 solution are required to determine stability, and computer programs modified as indicated may be used to calculate these loads. This provides a new and exact technique for the calculation of subcritical and supercritical flutter modes.

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Dynamic Characteristics of Rotor Blades: Integrating Matrix Method

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Nomenclature

EI = flapwise bending stiffness

e = distance between mass and elastic axis, positive when mass lies ahead of elastic center

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GJ = torsional stiffness
k_m^2 $= k_m^2 + k_{m2}^2$
$k_{m1,2}$ = mass radii of gyration about major neutral axis and about an axis perpendicular to chord through the elastic axis, respectively
m = mass per unit length
T = tension in the blade
w = amplitude of simple harmonic flapwise deflection, positive up
x = spanwise coordinate
ϕ = amplitude of simple harmonic torsional deformation, positive leading edge upwards
Ω = angular velocity of rotation
ω = frequency of vibration

I. Introduction

THE integrating matrix is a means of numerically integrating a function that is expressed in terms of the values of the function at equal increments of the independent variable. Similarly, a differentiating matrix is a means of numerically differentiating a function that is expressed in terms of the values of the function at equal increments of the independent variable. They can be derived by expressing the function as a polynomial in the form of a Newton's forward difference interpolation formula. The integrating matrix has been applied to obtain the natural frequencies of propeller blades for combined bending in two directions in Ref. 1.

In the present note this procedure is applied to determine the dynamic characteristics (natural frequencies and associated mode shapes) of rotor blades for coupled flapwise bending and torsion. This problem needs the introduction of a differentiating matrix in addition to the integrating matrix. Further, the following aspects of this technique which were not brought up in Ref. 1 are investigated. a) The accuracy of the mode shapes obtained by this technique when coupled degrees of freedom are involved; for example, coupled bending torsion, and bending in two directions, etc.; b) applicability to semidefinite systems; and c) orthogonality of eigenvectors.

II. Formulation of the Problem

The differential equations of motion for non-uniform rotor blades for coupled flapwise bending and torsion are derived in Ref. 2. They are given below for simple harmonic free vibration with frequency ω.

$$(EIw'')'' - (Tw')' - (\Omega^2 mxe\phi)' - \omega^2(mw + me\phi) = 0 \quad (1)$$

$$-(GJ\phi')' + \Omega^2 mxew' + \Omega^2 m(k_{m_2}^2 - k_{m_1}^2)\phi$$
$$-\omega^2(mew + mk_m^2\phi) = 0 \quad (2)$$

$$T' + \Omega^2 mx = 0 \quad (3)$$

In matrix notation these equations can be written as

$$([EI]\{w''\})'' - ([T]\{w'\})' - \Omega^2([me][x]\{\phi\})'$$
$$-\omega^2[m]\{w\} - \omega^2[me]\{\phi\} = \{0\} \quad (4)$$

$$-([GJ]\{\phi\})' + \Omega^2[me][x][D]\{w\}$$
$$+\Omega^2[m(k_{m_2}^2 - k_{m_1}^2)]\{\phi\} - \omega^2[me]\{w\}$$
$$-\omega^2[mk_m^2][x]\{1\} = \{0\} \quad (5)$$

$$\{T'\} + \Omega^2[m][x]\{1\} = \{0\} \quad (6)$$

The matrix $[D]$ in Eq. (5) represents the differentiating matrix.

The boundary conditions for the rotor which is fixed at the root are:

$$w_0 = 0; \quad w_0' = 0; \quad \phi_0 = 0; \quad (EIw'')_n = 0$$

$$((EIw'')' - Tw' - \Omega^2 mxe\phi)_n) = 0; \quad (GJ\phi')_n = 0; \quad T_n = 0 \quad (7)$$

The subscript zero denotes the fixed end of the blade and n denotes the free end of the blade.

Integrating Eqs. (4)-(6) by using the procedure of Ref. 1 yields the following eigenvalue problem:

$$[B]^{-1}[A]\{\alpha\} = \lambda\{\alpha\} \quad (8)$$

where

$$\lambda = \frac{1}{\omega^2}; \quad [A] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]; \quad [B] = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

$$[A_1] = [I]^2[EI]^{-1}[F]^2[m];$$

$$[A_2] = [I]^2[EI]^{-1}[F]^2[me]$$

$$[A_3] = [I][GJ]^{-1}[F][me];$$

$$[A_4] = [I][GJ]^{-1}[F][mk_m^2]$$

$$[B_1] = [1] + \Omega^2[I]^2[EI]^{-1}([F]$$
$$[m][x] - [P])$$

$$[B_2] = \Omega^2[I]^2[EI]^{-1}[F][me][x]$$

$$[B_3] = \Omega^2[I][GJ]^{-1}[F][me][x][D]$$

$$[B_4] = [1] + \Omega^2[I][GJ]^{-1}[F][m(k_{m_2}^2 - k_{m_1}^2)]$$

III. Numerical Results and Discussion

Natural frequencies for a non-rotating blade for coupled flapwise bending and torsion are computed by the above formulation for a number of stations, 5, 11, and 15, and are compared with the results obtained by the transmission matrix method (Ref. 3) in Table 1. Associated model shapes corresponding to 11 stations are presented in Table 2 and compared with the results obtained by the transmission matrix method.

It can be observed from Table 1 that integrating and differentiating matrix techniques yield accurate natural frequencies at least up to the first five for 11 stations. The accuracy improves with the increasing number of stations as expected. When the number of stations are five, we obtain three frequencies with reasonable accuracy. The third frequency is more accurate than the second frequency because the second frequency corresponds to the second bending mode and the third frequency corresponds to the first torsional

Table 1 Comparison of natural frequencies, coupled flapwise bending and torsion, fixed-free[a]

	Frequencies, rad/sec			
Mode no.	IM 5	IM II	IM 15	TM (Ref. 3)
1	31.05	31.05	31.05	31.05
2	189.37	193.73	193.74	193.74
3	390.80	390.87	390.87	390.87
4	578.93	539.21	539.54	539.56
5	1168.22	1042.95	1043.94	1041.74

[a] $R = 40.0$ in., $b_0 = 2.0$ in., $EI_1 = 25{,}000$ lb/in.^{2}, $GJ = 9{,}000$ lb/in.^{2}, $e = 0.4$ in., $k_{m_1}^2 = 0.18$ in.^{2}, $k_{m_2}^2 = 0.71$ in.^{2}, $\Omega = 0$.

Table 2 Comparison of mode shapes, coupled flapwise, bending and torsion, fixed-free

I Mode

	IM, 11		TM	
$\bar{x}$	w	ϕ	w	ϕ
0.0	0.0000	0.0000	0.0000	0.0000
0.2	0.0639	0.0081	0.0639	0.0014
0.4	0.2299	0.0159	0.2299	0.0026
0.6	0.4611	0.0227	0.4611	0.0038
0.8	0.7255	0.0276	0.7255	0.0046
1.0	1.0000	0.0295	1.0000	0.0049

II Mode

	IM, 11		TM	
$\bar{x}$	w	ϕ	w	ϕ
0.00	0.0000	0.0000	0.0000	0.0000
0.2	−0.2952	−0.2020	−0.2950	−0.0337
0.4	−0.6681	−0.3499	−0.6678	−0.0583
0.6	−0.5712	−0.3874	−0.5710	−0.0646
0.8	0.0817	−0.3299	0.0817	−0.0550
1.0	1.0000	−0.2791	1.0000	−0.0465

III Mode

	IM, 11		TM	
$\bar{x}$	w	ϕ	w	ϕ
0.0	0.0000	0.0000	0.0000	0.0000
0.2	−0.0060	0.3093	−0.0361	0.3093
0.4	−0.0207	0.5860	−0.1242	0.5860
0.6	−0.0324	0.8057	−0.1946	0.8057
0.8	−0.0327	0.9493	−0.1962	0.9493
1.0	−0.0257	1.0000	−0.1541	1.0000

mode. In the above computations the integrating matrix corresponding to third-order polynomial and differentiating matrix corresponding to fourth-order polynomial are used. The integrating matrix is derived in Ref. 1 and the differentiating matrix can be obtained from the formulae given in Ref. 4. It can be seen from Table 2 that I and II are predominantly bending modes and III is predominantly a torsional mode. It is evident from this table that bending deflections are accurate for predominantly bending modes and torsional deflections are accurate for predominantly, torsional modes. Therefore the integrating and differentiating matrix method does not yield accurate deflections for non-predominant degrees of freedom in a coupled mode shape, the reason being that this approach yields a non-self-adjoint eigenvalue problem even though the original system is self-adjoint. This can be observed from the eigenvalue problem given by Eq. (8), that neither $[A]$ nor $[B]$ are symmetric matrices; consequently, the modal vectors defined by the eigenvalue problem are not orthogonal in the normal sense. To solve for the eigenvalues, the QR-transformation method (Ref. 5) is preferable rather than the conventional sweeping and iteration technique since the modal vectors are not orthogonal in the normal sense. Even though the original continuous system has orthogonality properties for its eigenfunctions, they cannot be used to solve Eq. (8), since any approximate formulation must be self-contained. But one could use biorthogonality relationships, which involve solutions of two eigenvalues problems: (a) the original eigenvalue problem, and (b) the adjoint eigenvalue problem. Since the present eigenvalue problem is non-self-adjoint, it is preferable to use the complex QR-transformation to look for the possibility of complex eigenvalues. However, such values are not encountered in the present computation, which is also similar to problem discussed in Ref. 1.

The present method breaks down when applied to a system containing rigid-body degrees of freedom. To illustrate this problem, the method is applied to a uniform continuous beam which is hinged at one end and free at the other end. The equation of motion for simple harmonic free vibration with frequency ω is given by

$$w'''' - \lambda w = 0$$

where

$$\lambda = m\omega^2/EI$$

Integrating this four times successfully yields the following equation:

$$\{w\} - \lambda[I]^4\{w\} + [I]^3\{k_1\} + [I]^2\{k_2\} + [I]\{k_3\} + \{k_4\} = \{0\}$$

By applying the boundary conditions ($w_0 = w_0'' = w_n'' = w_n''' = 0$) to determine the constants of integration $\{k_1\}$ $\{k_2\}$ $\{k_3\}$ and $\{k_4\}$ one can observe that $\{k_3\}$ will be left undetermined and $\{k_2\}$ will have two different values. The reason for the failure of this technique for semidefinite systems may be that this approach is leading to a flexibility type of formulation (in the sense that the eigenvalue $\lambda = 1/\omega^2$) and such a formulation cannot exist for a semidefinite system.

In spite of the failure of this technique to semidefinite systems and inaccurate non-predominant modes in coupled problems, the method is highly appealing because of the advantages discussed in Ref. 1.

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Consistent Integral Thickness Utilization for Boundary Layers with Transverse Curvature

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Nomenclature

A = area
a = coefficient in Eqs. (16), $\bar{T}_w$
b = coefficient in Eqs. (16), $(\bar{T}_{aw} - \bar{T}_w)\epsilon$
c = coefficient in Eqs. (16), $1.0 - \bar{T}_{aw}$
C_f = skin friction coefficient

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